

Dequantization of Noncommutative Spaces and Dynamical Noncommutative Geometry

Freddy Van Oystaeyen
 Department of Mathematics & Computer Science
 University of Antwerp
 Middelheimcampus
 Middelheimlaan 1, 2020 Antwerpen, Belgium
 e-mail : fred.vanoystaeyen@ua.ac.be

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Abstract

The purely mathematical root of the dequantization constructions is the quest for a sheafification needed for presheaves on a noncommutative space. The moment space is constructed as a commutative space, approximating the noncommutative space appearing as a dynamical space, via a stringwise construction. The main result phrased is purely mathematical, i.e. the noncommutative stalks of some sheaf on the noncommutative space can be identified to stalks of some sheaf associated to it on the commutative geometry (topology) of the moment space. This may be seen as a (partial) inverse to the deformation–quantization idea, but in fact with a much more precise behaviour of stalks of sheaves. The method, based on minimal axiomatics necessary to rephrase continuity principles in terms of partial order (noncommutative topology) exclusively, leads to the appearance of objects like strings and (M–)branes. Also, spectral families and observables may be defined and studied as separated filtrations on the noncommutative topology! We highlight the relation with pseudo–places and generalized valuation theory. Finally, we hint at a new notion of “space” as a dynamical system of noncommutative topologies with the same commutative shadow (4 dimensional space–time if you wish) and variable (but isomorphic) moment spaces, which for special choices may be thought of as a higher dimensional brane–space.

Introduction

Noncommutative spaces obtained by quantization–deformation may to some extent be traced back to the base space they have been deformed from. A noncommutative topology in the sense of [VO1] has a so-called commutative shadow

that turns out to be a lattice. However, the lack of geometric points in a noncommutative space contrasts the set-theory based commutative geometry with its function theory and corresponding analysis. Looking for a converse construction for quantization, a better commutative approximation of a noncommutative space is needed; this construction should allow tracing of physical aspects like observables, spectral families etc...

Different notions of noncommutative space have been introduced e.g. noncommutative manifolds [Co], noncommutative “quantized” algebras [ATV], [VdB], general quantization-deformations [Ko]. In these theories the actual geometric object is left as a virtual object and results deal usually with noncommutative algebras or categories thought of as rings of functions or categories of modules.

In the author’s approach, started in [VO1], a noncommutative topology is at hand and then sheaf theory should to some extent replace function theory. Applied to Physics this model seems to allow some explanation of quantum phenomena but for explicit calculations and geometric reasoning a commutative model is unavoidable because at some point coordinatization and the use of real or complex numbers becomes necessary. In the model we propose these commutative techniques may be carried out in the dequantized space. The construction of this commutative approximation of noncommutative space derives from a dynamical version of noncommutative geometry that may also be seen as a noncommutative topological blow-up and blow-down construction. Restricting assumptions to bare necessities one finds that all structural properties only depend on a few intuitive axioms at the level of ordered (partial) structures. Respecting this philosophy we started from a totally ordered set (not even necessarily a group), T and suitably connected noncommutative spaces $\Lambda_t, t \in T$. A stringwise construction then yields a new spectral space, called the **moment space**, at each $t \in T$, with its topology in the classical sense i.e. a commutative space.

Sheaves on lattices were already termed quantum sheaves, but now we view (pre-)sheaves on noncommutative topologies fitting together in a kind of dynamical geometry and connect them to sheaves on the moment spaces. The idea that commutative moment spaces are good approximations of noncommutative spaces is reflected in the main result : the stalks at points of the moment space at $t \in T$ equal stalks at a point at some $\Lambda_{t'}, t' \in T$ for some t' in a prescribed closed T -interval containing t . The philosophy here is that there may not exist enough geometric points in the noncommutative geometry (Λ_t) at $t \in T$ but it holds in the moment space at t because it encodes dynamical information in some T -interval containing t . The construction is abstract and rather general, but when looking for applications in Physics one is only interested in one case ... reality ? Then one would think of T as a multidimensional irreversible “time” with “dimension” big enough to make up for the missing geometric points in the noncommutative space Λ_t at $t \in T$, i.e. the T -dimension explains the difference in dimension between the noncommutative space, or the moment space, and that of its space of points (the commutative shadow). Whereas the commutative shadow represents the abstract space where the mathematics takes place, e.g. space-time, the moment space represents the space where the mathematics

of observed and measured items takes place, and because observing or measuring takes time the latter space has to encode dynamical data. Stated as a slogan : “Observation Creates Space from Time”. Moreover, when observing a smallest possible entity, we would like to think of this as an observed point i.e. a point in the moment space, at t say. Tracing this via the dynamical noncommutative geometry to the commutative shadow (thinking of this as the fixed space we calculate in) the observed point appears as a string (it may be open or closed). On the other hand the point in the moment space is via a string of temporal points (not geometric points) in the noncommutative space also traceable to a possibly “higher dimensional string” in the moment space, in fact a tube-like string connecting opens in (the topologies of the noncommutative spaces blown down to) the spectral topology of the moment space (also identified to one commutative space for example). Of course this suggests that a version of M -theory already appears in the mathematical formalism explaining the transfer between commutative shadow and moment space.

Moreover the uncertainty principle also marks the difference between the two commutative spaces, in some sense expressing a quantum-commutativity of the noncommutative spaces, for example phrased in sheaf theoretical terms.

We have aimed at a rather self contained presentation of basic facts and definitions of the new mathematical objects in particular ... generalized Stone spaces and noncommutative topology.

The methods of this paper can be adapted to other types of noncommutative spaces e.g. quantales, Further generalization, e.g. to noncommutative Grothendieck topologies, has not been included, this would be an exercise using some material from [VO1]. We include some comments about specific examples and further mathematical elaborations using deformations of affine spaces or other nice classical (algebraic) varieties is left as work in progress.

1 Preliminaries on Noncommutative Topology

We consider (partially ordered) Λ with 0 and 1. When Λ is equipped with operations \wedge and \vee , we say that Λ is a noncommutative topology if the following axioms hold :

- i) for $x, y \in \Lambda$, $x \wedge y \leq x$ and $x \wedge y \leq y$ and $x \wedge 1 = 1 \wedge x = x$, $x \wedge 0 = 0 \wedge x = 0$, moreover $x \wedge \dots \wedge x = 0$ if and only if $x = 0$.
- ii) For $x, y, z \in \Lambda$, $x \wedge y \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z)$, and if $x \leq y$ then $z \wedge x \leq z \wedge y$, $x \wedge z \leq y \wedge z$.
- iii) Properties similar to i. and ii. with respect to \vee , in particular $x \vee x \vee \dots \vee x = 1$ if and only if $x = 1$.
- iv) Let $\text{id}_\wedge(\Lambda) = \{\lambda \in \Lambda, \lambda \wedge \lambda = \lambda\}$; for $x \in \text{id}_\wedge(\Lambda)$ and $x \leq z$ we have : $x \vee (x \wedge z) \leq (x \vee x) \wedge z$, $x \vee (z \wedge x) \leq (x \vee z) \wedge x$.

- v) For $x \in \Lambda$ and $\lambda_1, \dots, \lambda_n \in \Lambda$ such that $1 = \lambda_1 \vee \dots \vee \lambda_n$, we have

$$x = (x \wedge \lambda_1) \vee \dots \vee (x \wedge \lambda_n) = (\lambda_1 \wedge x) \vee \dots \vee (\lambda_n \wedge x).$$

There are left (right) versions of this definition as introduced in [VO.1]., but we do not go into this here. In fact we restrict attention to the situation where \vee is a commutative operation and Λ is \vee -complete i.e. for an arbitrary family \mathcal{F} of elements in Λ , $\vee \mathcal{F}$ exists in Λ , where $\vee \mathcal{F}$ is characterized by the property : $\lambda \leq \vee \mathcal{F}$ for all $\lambda \in \mathcal{F}$ and if $\lambda \leq \mu$ for all $\lambda \in \mathcal{F}$ then $\vee \mathcal{F} \leq \mu$.

In case only v) is dropped we call Λ a **skew topology**; this is sometimes interesting e.g. the lattice $L(H)$ of a Hilbert space H is a skew topology and condition v) does not hold ! Here $L(H)$ is the lattice of closed linear subspaces of H with respect to intersection and closure of the sum. The restriction to abelian \vee (most interesting examples are like this) entails that $\text{id}_\Lambda(\Lambda)$ is closed under the operation \vee . Moreover on $\text{id}_\Lambda(\Lambda)$ we introduce a new operation $\dot{\wedge}$ defined by $\sigma \dot{\wedge} \tau = \vee \{\gamma \in \text{id}_\Lambda(\Lambda), \gamma \leq \sigma \wedge \tau\}$ for $\sigma, \tau \in \text{id}_\Lambda(\Lambda)$. Let us write $SL(\Lambda)$ for the set $\text{id}_\Lambda(\Lambda)$ with the operations $\dot{\wedge}$ and \vee ; then $SL(\Lambda)$ is easily checked to be again a skew topology (with \vee commutative and being \vee -complete).

1.1 Lemma (cf. [VO.1] or [VO.2] 2.2.3. and 2.2.5)

If Λ is a skew topology \vee -complete with respect to a commutative operation \vee , then $SL(\Lambda)$ is a lattice satisfying the modular inequality.

A subset $X \subset \Lambda$ is **directed** if for every $x, y \in X$ there is a $z \in X$ such that $z \leq x, z \leq y$; we say that X is a **filter** in Λ if it is directed and for $x \in X, x \leq y$ yields $y \in X$. Two directed sets A and B in Λ are equivalent if they define the same filter $\overline{A} = \overline{B}$ where for any directed set A we put $\overline{A} = \{\lambda \in \Lambda, \text{there is an } a \in A \text{ such that } a \leq \lambda\}$. Let $\mathcal{D}(\Lambda)$ be the set of directed subsets in Λ and we write $A \sim B$ if the directed subsets A and B are equivalent, we write $[A]$ for the equivalence class of A and let $C(\Lambda)$ be the set of classes of directed subsets of Λ . We introduce a partial order in $C(\Lambda)$ by putting $A \leq B$ if $\overline{B} \subset \overline{A}$ and $[A] \leq [B]$ induced by the partial order on $\mathcal{D}(\Lambda)$ defined by the foregoing. For A and B in $\mathcal{D}(\Lambda)$ define $A \dot{\wedge} B = \{a \dot{\wedge} b, a \in A, b \in B\}$, $A \dot{\vee} B = \{a \dot{\vee} b, a \in A, b \in B\}$ and : $[A] \dot{\wedge} [B] = [A \dot{\wedge} B]$, $[A] \dot{\vee} [B] = [A \dot{\vee} B]$.

1.2 Lemma

If Λ is a skew topology, resp. a noncommutative topology, then so is $C(\Lambda)$ with respect to the partial order and operations $\dot{\wedge}$ and $\dot{\vee}$ as defined above. The canonical map $\Lambda \rightarrow C(\Lambda), \lambda \mapsto [\{\lambda\}]$ is a map of skew topologies.

We simplify notations by writing $[\{\lambda\}] = [\lambda]$ and call such an element **classical** in $C(\Lambda)$.

A directed set A in Λ is **idempotently directed** if for every $a \in A$ there exists an $a' \in A \cap \text{id}_\Lambda(\Lambda)$ such that $a' \leq a$; in this case $[A] \in i_\Lambda(C(\Lambda))$ but these elements of $C(\Lambda)$ may be thought of as obtained from a directed set having a cofinal subset of “commutative” opens (the idempotents belonging to the commutative shadow $S((\Lambda))$). We write $\text{Id}_\Lambda(C(\Lambda))$ for the subset of $\text{id}_\Lambda(C(\Lambda))$

consisting of the classes of idempotently directed subsets of Λ ; the elements $[A]$ of $\text{Id}_\wedge(\text{Id}_\wedge(C(\Lambda)))$ are called **strongly idempotents**. We identify Λ and the image of $\Lambda \rightarrow C(\Lambda)$, then observe that $\text{Id}_\wedge(C(\Lambda)) \cap \Lambda = \text{id}_\wedge(\Lambda)$. We shall write $\prod(\Lambda)$ for the skew (noncommutative) topology obtained by talking \wedge -finite bracketed expression $P(\wedge, \vee, x_i)$ in terms of strong idempotents $x_i \in \text{Id}_\wedge(C(\Lambda))$; similarly we write $T(\Lambda)$ for the skew (noncommutative) topology obtained by taking \wedge -finite bracketed expressions $p(\wedge, \vee, \lambda_i)$ in idempotents $\lambda_i \in \text{id}_\wedge(\Lambda)$.

It is not hard to verify : $\prod(\Lambda) = \prod(T(\Lambda))$. Moreover $C(T(\Lambda))$ satisfies the same axioms (i. to v.) as Λ and $T(\Lambda)$ but with respect to $\text{Id}_\wedge(C(\Lambda))$. The “strong” commutative shadow of \prod is obtained by restricting \wedge on $\text{id}_\wedge(C(\Lambda))$ to $\text{Id}_\wedge(C(\Lambda))$ and viewing $SL_s(\prod)$ as the lattice structure induced on $\text{Id}_\wedge(C(\Lambda))$.

1.3 Lemma

If Λ is a \vee -complete noncommutative topology such that \vee is commutative then : $SL_s(\prod) = C(SL(\Lambda))$.

1.4 Definition. Generalized Stone Topology

Consider a skew topology Λ and $C(\Lambda)$. For $\lambda \in \Lambda$, let $O_\lambda \subset C(\Lambda)$ be given by $O_\lambda = \{[A], \lambda \in \overline{A}\}$. It is very easy to verify : $O_{\lambda \wedge \mu} \subset O_\lambda \cap O_\mu$, $O_{\lambda \vee \mu} \supset O_\lambda \cup O_\mu$, hence the O_λ define a basis for a topology on $C(\Lambda)$ termed : generalized Stone topology. This definition obtains a more classical meaning when related to suitable point-spectra constructed in $C(\Lambda)$.

We say that $[A]$ in $C(\Lambda)$ is a **minimal point of Λ** if \overline{A} is a maximal filter, i.e. $\overline{A} \neq \Lambda$ but if $\overline{A} \subsetneq B$ where B is a filter then $B = \Lambda$; it follows that a minimal point is necessarily in $\text{id}_\wedge(C(\Lambda))$ and it is indeed a minimal nonzero element of the poset $C(\Lambda)$. An **irreducible point** $[A]$ of Λ is characterized by either one of the following equivalent properties :

- i) $[A] \leq \vee\{[A_\alpha], \alpha \in \mathcal{A}\}$ yields $[A] \leq [A_\alpha]$ for some $\alpha \in \mathcal{A}$.
- ii) If $\vee\{\lambda_\alpha, \alpha \in \mathcal{A}\} \in \overline{A}$ then $\lambda_\alpha \in \overline{A}$ for some $\alpha \in \mathcal{A}$.

More general types of points may be considered, e.g. the elements of a so-called quantum-basis, cf [VO.2], but we need not go into this here. Under some suitable condition often (but not always) present in examples the irreducible points in $\text{Id}_\wedge(C(\Lambda))$ are exactly those that are \vee -irreducible in $C(\Lambda)$ (e.g. if Λ satisfies the weak FDI property, cf. [VO.1], Proposition 5.9.).

Define the (irreducible) **point-spectrum** by putting : $\text{Sp}(\Lambda) = \{[p], [p] \text{ an (irreducible) point of } \Lambda\}$. Put $p(\lambda) = \{[p] \in \text{Sp}(\Lambda), [p] \leq [\lambda]\}$ for $\lambda \in \Lambda$, then $p(\lambda \wedge \mu) \subset p(\lambda) \cap p(\mu)$, $p(\lambda \vee \mu) = p(\lambda) \cup p(\mu)$. Thus the $p(\lambda)$ define a basis for a topology on $\text{Sp}(\Lambda)$ called the point-topology. Write $SP(\Lambda)$ for $\text{Sp}(\Lambda) \cap \text{Id}_\wedge(C(\Lambda))$ and refer to this as the **Point-spectrum** (capital P). For $\lambda \in \Lambda$ we consider $P(\lambda) = \{[P] \in \text{SP}(\Lambda), [P] \leq [\lambda]\}$ and this induces the Point-topology on $\text{SP}(\Lambda)$; this time we even have $P(\lambda \wedge \mu) = P(\lambda) \cap P(\mu)$ and this time we even have $P(\lambda \vee \mu) = P(\lambda) \cup P(\mu)$ and this defines a topology on

$\text{SP}(\text{SL}(C(\Lambda)))$. Similar constructions may be applied to the minimal point-spectrum $Q\text{Sp}(\Lambda)$ on $\text{Sp}(\Lambda)$ the generalized Stone topology is nothing but the point-topology. In the foregoing Λ may be replaced by $T(\Lambda)$ or $\prod(\Lambda)$ with topologies induced by the generalized Stone topology on the point spectra always again being called : generalized Stone topology. Finally the generalized Stone topology can also be defined on the commutative shadow $\text{SL}(\Lambda)$, which is a modular lattice, and then the topology induced on $Q\text{SP}(\text{SL}(\Lambda))$ is exactly the Stone topology of the Stone space of $\text{SL}(\Lambda)$.

In the special case $\Lambda = L(H)$ (only a skew topology) the generalized Stone space defined on $Q\text{SP}(L(H))$ is exactly the classical Stone space as used in Gelfand duality theory for $L(H)$. Warning : $L(H)$ does not satisfy the axiom v . and whereas $Q\text{SP}(L(H))$ is rather big, $\text{Sp}(L(H)) = \text{SP}(L(H))$ is empty ! Moreover over $L(H)$ there are no sheaves but there will be many sheaves over $C(L(H))$ making sheafification of a separated presheaf over $L(H)$ possible over $Q\text{SP}(L(H))$.

The foregoing fixes a context for results in the sequel, however the methods in Sections 2-5 are rather generic and can be applied to other notions of noncommutative spaces.

2 Dynamical Noncommutative Topology

The realization of a relation between the static and the dynamic goes far back, at least to D'Alembert but the notion that the first may be used to study the latter may be not completely well founded. Almost existentialistic problems defy the correctness of a mathematical description of so-called physical reality. So we choose for a converse approach constructing space as a dynamical noncommutative topological space and defining geometrical objects as existing over some parameter-intervals. Noncommutative continuity is introduced via the variation of an external parameter in a totally ordered set T (if one wants to consider this as a kind of time, fine... but then this time is an index, not a dimension). This is philosophically satisfying because momentary observations which are only abstractly possible (real measurement takes time) put us in the discrete-versus-continuous situation of noncommutative geometry !

Let T be a totally ordered set and for every $t \in T$ we give a noncommutative space Λ_t . This can have several meanings, in the sequel we take this to mean that Λ_t is the generalized Stone space constructed on $C(X_t)$ for some skew topology X_t as in Section 1. This is just to fix ideas, in fact one could just as well restrict to topologies induced by the generalized Stone topology on point spectra of any type, see also Section 1 or [VO1], or take pattern topologies as introduced in [VO1,2], or go to other theories and take quantales etc... For $t \leq t'$ in T we have $\varphi_{tt'} : \Lambda_t \rightarrow \Lambda_{t'}$ which are poset maps respecting \wedge and \vee ; when $t = t'$ we take $\varphi_{tt} = 1_{\Lambda_t}$ to be the identity of Λ_t and when $t \leq t' \leq t''$ then we demand that $\varphi_{t't''} \circ \varphi_{tt'} = \varphi_{tt''}$, where notation for composition of maps is conventional i.e. writing the one acting last first. If $A_t \subset \Lambda_t$ is a directed set then $\varphi_{tt'}(A_t) \subset \Lambda_{t'}$, for $t \leq t'$, is again a directed set.

It is easily verified that if we start from a system $\{X_t, \Psi_{tt'}, T\}$ defined as above, we obtain a similar system $\{C(X_t), \Psi_{tt'}^e, T\}$ where $\Psi_{tt'}^e : C(X_t) \rightarrow C(X_{t'})$ is given by putting : $\Psi_{tt'}^e([A]) = [\Psi_{tt'}(A)]$, for $[A] \in C(X_t)$ and $t \in t'$ in T . In case it is interesting to view Λ_t as coming from some X_t via $C(X_t)$ we may restrict attention to systems given by $\varphi_{tt'}, t \leq t'$ in T , stemming from $\psi_{tt'}$ on X_t by extension as indicated above. Note that not every system $\{C(X_t), \varphi_{tt'}, T\}$ has to derive from a system $\{X_t, \psi_{tt'}, T\}$ in general.

2.1 Lemma

Any system of poset maps $\varphi_{tt'}, t \leq t'$ in T , defines a system of poset maps $\varphi_{tt'}^e, t \leq t'$ in T . If the maps $\varphi_{tt'}$ respect the operations \wedge and \vee in the Λ_t then so does $\varphi_{tt'}^e$ for $C(\Lambda_t), t \in T$. In this situation $\varphi_{tt'}$ maps \wedge -idempotent elements of Λ_t to \wedge -idempotent elements of $\Lambda_{t'}$ (also \vee -idempotent to \vee -idempotent) moreover if $[A_t]$ is strongly idempotent in $C(\Lambda_t)$ then $[\varphi_{tt'}(A_t)]$ is a strongly idempotent element of $C(\Lambda_{t'})$, for every $t \leq t'$ in T ,

Proof First statements follow obviously from : for directed sets A and B ,

$$\begin{aligned} \varphi_{tt'}^e([A] \wedge [B]) &= \varphi_{tt'}^e([A \wedge B]) = [\varphi_{tt'}(A \wedge B)] \\ &= [\varphi_{tt'}(A) \wedge \varphi_{tt'}(B)] = [\varphi_{tt'}(A)] \wedge [\varphi_{tt'}(B)] \end{aligned}$$

for $t \leq t'$ in T . Similar with respect to \vee , using $\dot{\vee}$. In case $\lambda \in \Lambda_t$ is idempotent in Λ_t then $\varphi_{tt'}(\lambda) \wedge \varphi_{tt'}(\lambda) = \varphi_{tt'}(\lambda \wedge \lambda) = \varphi_{tt'}(\lambda)$ for $t \leq t'$ in T . Finally if A is idempotently directed look at $\varphi_{tt'}(a)$ for $a \in A_t$; by assumption there exists some $\mu \in \text{id}_\wedge(\Lambda_t)$ such that $\mu \leq a$, thus $\varphi_{tt'}(\mu) \leq \varphi_{tt'}(a)$ and $\varphi_{tt'}(\mu) \in \text{id}_\wedge(\varphi_{tt'}(A_t))$, for $t \leq t'$ in T . Consequently $\varphi_{tt'}(A_t)$ is idempotently directed too.

The skew topology \prod_t , introduced after Lemma 1.2. is called the pattern topology of X_t , i.e. it is obtained by taking all \wedge -finite bracketed expressions with respect to \vee and \wedge in the letters of $\text{Id}_\wedge(C(\Lambda_t))$.

2.2 Corollary

The system $\{\Lambda_t, \varphi_{tt'}, T\}$ induces a system $\{\prod_t, \varphi_{tt'} | \prod_t, T\}$, satisfying the same properties, on the pattern topologies.

In general the $\varphi_{tt'}, t \leq t'$, do not map poits of Λ_t to points of $\Lambda_{t'}, t \leq t'$, neither does $\varphi_{tt'}$ respect the operation $\dot{\wedge}$ of the commutative shadow $SL(\Lambda_t)$, i.e. the $\varphi_{tt'}$ do not necessarily induce a system on the commutative shadows !

2.3 Axioms for Dynamical Noncommutative Topology (DNT)

A system $\{\Lambda_t, \varphi_{tt'}, T\}$ is called a DNT if the following five conditions are satisfied :

- DNT.1 Writing 0, resp. 1, for the minimal, resp. maximal element of Λ_t (we shall assume these exist throughout) then $\varphi_{tt'}(0) = 0$, $\varphi_{tt'}(1) = 1$ for all $t \leq t'$ in T .
- DNT.2 For all $t \in T$, $\varphi_{tt} = 1_{\Lambda_t}$ and for $t \leq t' \leq t''$ in T , $\varphi_{t't''} \circ \varphi_{tt'} = \varphi_{tt''}$. Moreover, all $\varphi_{tt'}$ preserve \wedge and \vee . Hence DNT.1. and 2. just restate that $\{\Lambda_t, \varphi_{tt'}, T\}$ is as before.
- DNT.3 If for some $t \in T$, $0 < x < y$ in Λ_t , then there is a $t < t_1$ in T such that for $z_1 \in \Lambda_{t_1}$ satisfying $\varphi_{tt_1}(x) < z_1 < \varphi_{tt_1}(y)$ there is a $z \in \Lambda_t$, $x < z < y$, such that $\varphi_{tt_1}(z) = z_1$.
- DNT.4 For every $t \in T$ and $0 < x < z < y$ in Λ_t there exist $t_1, t_2 \in T$ such that $t_1 < t < t_2$ and for every $t' \in]t_1, t_2[$ we have either $t \leq t'$ and $\varphi_{tt'}(x) < \varphi_{tt'}(z) < \varphi_{tt'}(y)$, or $t' \leq t$ and then if $x' < y'$ in $\Lambda_{t'}$ exist such that $\varphi_{t't}(x') = x$, $\varphi_{t't}(y') = y$ then there also exist z' in $\Lambda_{t'}$ such that $x' < z' < y'$ and $\varphi_{t't}(z') = z$. Taking the special case $y = 1$ and $y' = 1$ then we see that a nontrivial strict relation in Λ_t is alive in an open T -interval containing t .
- DNT.5 **T -local unambiguity.** In the situation of DNT.3, resp. DNT.4, the $t_1 \in T$, resp. t_1 and t_2 , may be chosen such that $z \in \Lambda_t$ is the unique element such that $\varphi_{tt_1}(z) = z_1$, resp. x', y', z' in Λ_t , are the unique elements such that $\varphi_{t't}(z') = z$, $\varphi_{t't}(y') = y$, $\varphi_{t't}(x') = x$, or when $t \leq t'$ the x, y, z , are unique elements mapping to $\varphi_{tt'}(z)$, $\varphi_{tt'}(y)$, $y_{tt'}(x)$ resp.

Since we are able to take finite intersection of open T -intervals in the totally ordered set T , we may extend the foregoing to finite chains of $0 < x_1 < x_2 < \dots < x_n, n \geq 3$.

Observe that the axioms allow that non-interacting elements, i.e. x such that $0 < x < 1$ is the only orderrelation it is involved in, may appear and disappear momentarily. here disappearing means going to 0 under all $\varphi_{tt'}, t \leq t'$, if $x \in \Lambda_t$.

Very often properties studied are only preserved in some T -interval, in particular this happens often when trying to derive a property of a related system from another one that may be globally defined for T . This leads to an interesting phenomenon, already encoding some aspect of the moment-spaces to be defined later.

2.4 Definition of Observed Truth

A statement in terms of finitely many ingredients of a DNT and depending on parametrization by $t \in T$ is said to be an **observed truth** at $t_0 \in T$ if there is an open T -interval $]t_1, t_2[$ containing t_0 , such that the statement holds for parameter values in this interval.

It seems that mathematical statements about a DNT turn into “observed truth” when one tries to check them in the commutative shadow, meaning on the negative side that many global (over T) properties of a DNT cannot be established globally over T in the commutative world !

The noncommutative topologies Λ_t considered in the sequel will be such that \vee is commutative and \vee of arbitrary families exist; in fact one may restrict to so-called “virtual topologies” as introduced in [VO2]; here we do not need to assume axiom (ν) with respect to global covers, we may want to restrict to this case when needed. We refer to the special case mentioned as a DVT.

2.5 Proposition

Let $\{\Lambda_t, \varphi_{tt'}, T\}$ be a DVT and let $SL(\Lambda_t)$ be the commutative shadow of Λ_t with maps $\varphi_{tt'} : SL(\Lambda_t) \rightarrow SL(\Lambda_{t'})$, $t \leq t'$ in T , just being the restrictions of the $\varphi_{tt'}$ (using same notation). Then the statement that $\{SL(\Lambda_t), \varphi_{tt'}, T\}$ is a DVT too is an observed truth at every $t_0 \in T$.

Proof All $\varphi_{tt'}$ map \wedge -idempotents to \wedge -idempotents, cf. Lemma 2.1., so DNT.1. is obvious. For DNT.2 we have to check that $\varphi_{tt'}$ preserves \wedge on $\text{id}_\wedge(\Lambda_t)$, since $\vee = \vee$ now we have nothing to check for \vee . Look at $t_0 \in T$, $\varphi_{t_0t} : \Lambda_{t_0} \rightarrow \Lambda_t$ and σ, τ in $\text{id}_\wedge(\Lambda_{t_0})$. If $\sigma < \tau$ then $\varphi_{t_0t}(\sigma) \leq \varphi_{t_0t}(\tau)$ and $\varphi_{t_0t}(\sigma) \wedge \varphi_{t_0t}(\tau) = \varphi_{t_0t}(\sigma) = \varphi_{t_0t}(\sigma \wedge \tau)$, interchange the role of σ and τ in case $\tau < \sigma$. So we may assume σ and τ to be incomparable. Restricting t to a suitable T -interval (DNT 5) we may assume that $\varphi_{t_0t}(\sigma) \neq \varphi_{t_0t}(\tau)$. Assume $\varphi_{t_0t}(\sigma \wedge \tau) < \varphi_{t_0t}(\sigma) \wedge \varphi_{t_0t}(\tau)$.

If $\varphi_{t_0t}(\sigma \wedge \varphi_{t_0t}(\tau)) = \varphi_{t_0t}(\sigma)$ (a similar argument will hold if σ and τ are interchanged) then $\varphi_{t_0t}(\sigma) \leq \varphi_{t_0t}(\tau)$, hence $\varphi_{t_0t}(\sigma) < \varphi_{t_0t}(\tau)$. Using DNT.5 again, taking t close enough to t_0 , we obtain $\sigma \wedge \tau < \sigma_1 < \tau$ such that $\varphi_{t_0t}(\sigma \wedge \tau) < \varphi_{t_0t}(\sigma_1) = \varphi_{t_0t}(\sigma) < \varphi_{t_0t}(\tau)$. Passing to $[t_0, t]$ small enough in order to have unambiguity for $\varphi_{t_0t}(\sigma)$, we arrive at $\sigma_1 = \sigma$, contradicting the incomparability of σ and τ . Therefore we arrive at strict relations : $\varphi_{t_0t}(\sigma \wedge \tau) < \varphi_{t_0t}(\sigma) \wedge \varphi_{t_0t}(\tau) < \varphi_{t_0t}(\sigma), \varphi_{t_0t}(\tau)$. We may moreover assume (DNT.3) that t is close enough to t_0 so that there is a $z \in \Lambda_{t_0}$ such that $\sigma \wedge \tau < \sigma, \tau$ and $\varphi_{t_0t}(z) = \varphi_{t_0t}(\sigma) \wedge \varphi_{t_0t}(\tau)$. If z is not \wedge -idempotent, then $\sigma \wedge \tau < z \wedge z < \sigma$ would lead to $\varphi_{t_0t}(z \wedge z) = \varphi_{t_0t}(\sigma) \wedge \varphi_{t_0t}(z)$ because φ_{t_0t} respects \wedge and the latter is idempotent in Λ_t ; then $\varphi_{t_0t}(z) = \varphi_{t_0t}(z \wedge z)$ but the unambiguity guaranteed by the choice of t close enough to t_0 (DNT.5) then yields $z = z \wedge z$ or $z \in \text{id}_\wedge(\Lambda_{t_0})$. Thus $z = \sigma \wedge \tau$ by definition, a contradiction. Consequently, for t in some small enough T -interval containing t_0 we have obtained : $\varphi_{t_0t}(\sigma \wedge \tau) = \varphi_{t_0t}(\sigma) \wedge \varphi_{t_0t}(\tau)$, thus DNT.2 is an observed truth. To check DNT.3, take $\sigma < \tau$ in $\text{id}_\wedge(\Lambda_{t_1})$, $t < t_1$ such that $z_1 \in \text{id}_\wedge(\Lambda_{t_1})$ exists such that we have $\varphi_{tt_1}(\sigma) < z_1 < \varphi_{tt_1}(\tau)$. Now by DNT.3. for $\{\Lambda_t, \varphi_{tt'}, T\}$ there is a $z \in \Lambda_t, z < \tau$, such that $\varphi_{tt_1}(z) = z_1$, and DNT.5 for $(\Lambda_t, \varphi_{tt'}, T)$, used as in foregoing part of the proof, yields $\varphi_{tt_1}(z) = z_1$ with z also \wedge -idempotent in Λ_t , for t_1 close enough to t . The proof of DNT.4 follows in the same way and DNT.5 is equally obvious because unambiguity in a suitable T -interval allows to pull back idempotency. Therefore all DNT-axioms hold for $\{SL(\Lambda_t), \varphi_{tt'}, T\}$ in a suitable T -interval, hence we have the observed truth statement that $\{SL(\Lambda_t), \varphi_{tt'}, T\}$ is DNT. \square

Now fix a notion of point i.e. either minimal point or irreducible point as in

Section 1. We say that $\lambda_t \in \Lambda_t$ is a **temporal point** if $t \in]t_0, t_1[$ such that for some $t' \in]t_0, t_1[$ there is a point $p_{t'} \in \Lambda_{t'}$ such that : either $t \leq t'$ and $\varphi_{tt'}(\lambda_t) = p_{t'}$, or $t' \leq t$ and $\varphi_{t't}(\lambda_t) = p_{t'}$; in the first case we say λ_t is a **future point**, in the second case a **past point**. The system $\{\Lambda_t, \varphi_{tt'}, T\}$ is said to be **temporally pointed** if for every $t \in T$ and $\lambda_t \in \Lambda_t$ there exists a family of temporal points $\{p_{\alpha, t}; \alpha \in \mathcal{T}\}$ in Λ_t such that λ_t is covered by it, i.e. $\lambda_t = \vee \{p_{\alpha, t}, \alpha \in \mathcal{A}\}$. Write $T\mathcal{P}(\Lambda_t)$ for the set of temporal points of Λ_t , if we write $\text{Spec}(\Lambda_t) = \{p_{t'} \text{ point in } \Lambda_{t'}, p_{t'} \text{ defines a temporal point of } \Lambda_t\}$ then $T\mathcal{P}(\Lambda_t)$ may also be written as $T\text{Spec}(\Lambda_t)$ (note $T\text{Spec}(\Lambda_t)$ is in Λ_t but $\text{Spec}(\Lambda_t)$ not).

We need to build in more “continuity” aspects in the DVT-axioms without using functions or extra assumptions on T e.g. that it should be a group. A temporary pointed system $\{\Lambda_t, \varphi_{tt'}, T\}$ is a **space continuum** if the following conditions hold :

- SC.1 There is a minimal closed interval I_t containing t in T such that $T\text{Spec}(\Lambda_t)$ has support in I_t . The set of points in $\Lambda_{t'}$ with $t' \in I_t$, representing temporary points in Λ_t is then called the **minimal spectrum** for $T\text{Spec}(\Lambda_t)$, denoted by $\text{Spec}(\Lambda_t, I_t)$.
- SC.2 For any open T -interval I such that $I_t \subset I$ there exists an open T -interval I_t^* with $t \in I_t^*$, such that for $t' \in I_t^*$ we have $I_{t'} \subset I$.
- SC.3 For intervals $[t_1, t_2]$ and $[t_3, t_4]$ we write $[t_1, t_2] < [t_3, t_4]$ if $t_1 \leq t_3$ and $t_2 \leq t_4$ (similarly for open intervals). If $t \leq t'$ in T then $I_t < I_{t'}$. This provides an “orientation” of the variation of the minimal spectra !
- SC.4 **Local Preservation of Directed Sets** For given $t \leq t'$ in I_t and any directed set A_t in Λ_t , the subset $\{\gamma_t \in A_t, \text{ there exists } \xi_t < \gamma_t \text{ in } A_t \text{ such that } \varphi_{tt'}(\xi_t) < \varphi_{tt'}(\gamma_t)\}$ is cofinal in A_t (defines the same limit $[A_t]$). For $t'' \leq t$ in I_t there is a directed set $A_{t''}$ in $\Lambda_{t''}$ mapped by $\varphi_{t''t}$ to a cofinal subset of A_t .

A subset J of T is **relative open around** $t \in T$ if it is intersection of I_t and an open T -interval. For $x = (\dots, x_t, \dots) \in \prod_{t \in T} \Lambda_t$ we put $\text{sup}(x) = \{t \in T, x_t \neq 0\}$. We say that such an x is **topologically accessible** if all $x_{t'} \in \text{sup}(x)$, are classical i.e. $x_t = [\chi_t]$ (for some $\chi_t \in X_t$ and $\Lambda_t = C(X_t)$). In case we do not consider Λ_t as coming from some X_t the condition becomes void. An x as before is said to be **t -accessible** if $\text{sup}(x) = J$ is relative open around t and for all $t' \leq t''$ in J we have $\varphi_{t't''}(x_{t'}) \leq x_{t''}$. When Λ_t has enough points i.e. if $I_t = \{t\}$, then the points in an open for the point topology would be characterized by $\{p, p \leq [\chi_t]\} = U(\chi_t)$ for some $\chi_t \in X_t$. When Λ_t does not have enough points then we have to modify the definition of point spectrum and point topology correspondingly. If $x = (\dots, x_t, \dots)$ is t accessible and $p_{t'} \in \text{Spec}(\Lambda_t, I_t)$ then we say $p_{t'} \in x$ if $t' \in J = \text{sup}(x)$ and there exists an open T -interval $J_1 \subset J$ with $t' \in J_1$ such that for $t'' \in J_1$ we have : if $t' \leq t''$ then $p_{t''} = \varphi_{t't''}(p_{t'}) \leq x_{t''}$, or if $t'' \leq t'$ there is a $p_{t''} \in \Lambda_{t''}$ such that $\varphi_{t''t'}(p_{t''}) = p_{t'}, p_{t''} \leq x_{t''}$, i.e. $\{p_{t''}, t'' \in J_1\}$ is the restriction of a temporal

point representing $p_{t'}$ defined over a bigger T -interval $]t_0, t_1[$ containing both t' and t (note : J_1 need not contain t).

2.6 Theorem

The empty set together with the sets $U_t(x) = \{p_{t'}, p_{t''} \in x \text{ for some } t' \in I_t\} \subset \text{Spec}(\Lambda_t, I_t)$, x being t -accessible in $\prod_{t \in T} \Lambda_t$, form a topology on $\text{Spec}(\Lambda_t, I_t)$, called **spectral topology** at $t \in T$.

Proof Consider $x \neq y$ both t -accessible with respective T -intervals J , resp. J' contained in I_t . If $p_{t'} \in U_t(x) \cap U_t(y)$ then $t' \in J \cap J'$ and for every $t_1 \in J$, $p_{t_1} \leq x_{t_1}$, for every $t_2 \in J'$, $p_{t_2} \leq y_{t_2}$. Of course the interval $J \cap J'$ is relative open around t . If $t' \leq t''$ with $t'' \in J \cap J'$ then $p_{t''} = \varphi_{t't''}(p_{t'})$ is idempotent in $\Lambda_{t''}$ because $p_{t'}$ is in $\Lambda_{t'}$ as it is a point. Hence we obtain :

$$p_{t''} = p_{t''} \wedge p_{t''} \leq x_{t''} \wedge y_{t''}$$

Obviously for all $t'' \leq t'''$ in $J \cap J'$ we do have : $\varphi_{t''t'''}(x_{t''} \wedge y_{t''}) \leq x_{t'''} \wedge y_{t'''}$. On the other hand, for $t'' \leq t'$ we obtain : $\varphi_{t't''}(p_{t''}) = p_{t'}$ and therefore $p_{t'} \leq \varphi_{t't''}(x_{t''}) \leq x_{t'}$, as well as $p_{t'} \leq \varphi_{t't''}(y_{t''}) \leq y_{t'}$. Hence, again by idempotency of $p_{t'}$ in $\Lambda_{t'}$ we arrive at $p_{t'} \leq x_{t'} \wedge y_{t'}$. By restricting $J \cap J'$ to the interval obtained by allowing only those $t'' \leq t'$ which belong to an (open) unambiguity interval for $p_{t'}$ we arrive at a relative open around t , say $J'' \subset J \cap J'$, containing t' .

Now for $p_{t''}$ with $t'' \in J''$ it follows that $p_{t''}$ is idempotent because both $p_{t''}$ and $p_{t''} \wedge p_{t''}$ map to $p_{t'}$ via $\varphi_{t't''}$ for $t'' \leq t'$ (other t'' in J'' are no problem). Thus for t'' in J'' we do arrive at $p_{t''} \leq x_{t''} \wedge y_{t''}$. Define w by putting $w_{t''} = x_{t''} \wedge y_{t''}$ for $t'' \in J''$. Clearly, w is t -accessible and $p_{t'} \in U_t(w)$. Conversely if $p_{t'} \in U_t(w)$ then $p_{t'} \in U_t(x) \cap U_t(y)$ is clear because J'' used in the definition of w is open in $J \cap J'$. Now we look at a union of $U_{i,t} = U_t(x_i)$ for $i \in J$ and each x_i being A -accessible with corresponding relative open interval J_i in I_t . Define w over the “interval” $J = \cup_i \{J_i, i \in \mathcal{J}\}$ by putting $w_t = \vee \{x_{i,t}, i \in \mathcal{J}\}$ for $t \in J$. It is clear that J is relative open around t and for all $t_1 \leq t_2$ in J we have $\varphi_{t_1 t_2}(w_{t_1}) \leq w_{t_2}$ because $\varphi_{t_1 t_2}$ respects arbitrary \vee . Now $p_{t'} \in w$ means that $p_{t''} \leq \vee \{x_{i,t''}, i \in \mathcal{J}\}$ for t'' in some relative open containing t' , say $J_1 \subset J$. We use relative open sets in T because I_t was closed and there are two situations to consider concerning $t' \in I_t$. First if t' is the lowest element of I_t then for all $t'' \in J_1$ we have that $p_{t''} = \varphi_{t't''}(p_{t'}) \leq \varphi_{t't''}(\vee \{x_{i,t'}, i \in \mathcal{J}\})$ and for all $t' \leq t_1 \leq t''$ we also obtain : $p_{t_1} \leq \varphi_{t't_1}(\vee \{x_{i,t'}, i \in \mathcal{J}\})$ and $p_{t''} \leq \varphi_{t_1 t''}(\vee \{x_{t_1 t_1}, i \in \mathcal{J}\})$. Otherwise, if t' is not the lowest element of I_t then we may restrict J_1 to be an open interval $]t_0, t'_0[$ containing t' with $t_0 \in J$. The same reasoning as in the first case yields for all $t'' \in]t_0, t'_0[$ so that : $p_{t''} \leq \varphi_{t_0 t''}(\vee \{x_{i,t_0}, i \in \mathcal{J}\})$ and for any $t' \leq t_1 \leq t''$ $p_{t''} \leq \varphi_{t_1 t''}(\vee \{x_{i,t_1}, i \in \mathcal{J}\}) = \vee \{\varphi_{t_1 t''}(x_{i,t_1}), i \in \mathcal{J}\}$. Since $t' \in J_1$ we obtain $p_{t'} \leq \vee \{\varphi_{t_1 t'}(x_{i,t_1}), i \in \mathcal{J}\}$ for all $t_1 \in [t_0, t']$.

Since $p_{t'}$ is a point in $\Lambda_{t'}$ there is an $i_0 \in J$ such that $p_{t'} \leq \varphi_{t_0 t'}(x_{i_0, t_0})$ and therefore we have that $p_{t'} \leq \varphi_{t_1 t'}(x_{i_0, t_1})$ with $t_1 \in [t_0, t']$, the gain being that

i_0 does not depend on t_1 here ! Now for $t'' \geq t'$ in $J_1 \cap J_{i_0}$ (note that this is not empty because x_{i_0} is nonzero at t_0 because $p_{t'} \leq \varphi_{t_0 t''}(x_{i_0, t_0})$ would then make $p_{t'}$ zero and we do not look at the zero (the empty set) as a point of Λ_t) we obtain :

$$(*) \quad p_{t''} = \varphi_{t' t''}(p_{t'}) \leq \varphi_{t' t''}(x_{i_0, t'}) \leq x_{i_0, t''}$$

In the other situation $t'' \leq t'$ in $J_1 \cap J_{i_0}$ we have $\varphi_{t'' t'}(p_{t''}) = p_{t'}$, $\varphi_{t'' t'}(x_{i_0, t''}) \leq x_{i_0, t'}$. By restricting $J_1 \cap J_{i_0}$ further so that the $t'' \leq t'$ are only varying in an (open) unambiguity interval for $p_{t'}$, say $J_2 \subset J_1 \cap J_{i_0}$, we arrive at one of two cases : either $p_{t''} = x_{i_0, t''}$ or else $p_{t''} \neq x_{i_0, t''}$ and also $p_{t'} < x_{i_0, t'}$. In the first case $p_{t'} \in x_{i_0}$ follows because $p_{t_1} = \varphi_{t'' t_1}(x_{i_0, t''}) \leq x_{i_0, t_1}$ for t_1 in $]t'', 1[\cap J_2$, the latter interval containing t' is relative open again. In the second case we may look at $p_{t'} < x_{i_0, t'} < 1$, hence there exists a $z_{t''}$ such that $p_{t''} < z_{t''} < 1$ and $\varphi_{t'' t'}(z_{t''}) = x_{i_0, t'}$. Again we have to distinguish two cases, first $\varphi_{t'' t'}(x_{i_0, t''}) = x_{i_0, t'}$ or $\varphi_{t'' t'}(x_{i_0, t''}) < x_{i_0, t'}$.

In the first case $z_{t''}$ and $x_{i_0, t''}$ map to the same element via $\varphi_{t'' t'}$, hence up to restricting the interval further such that t'' stays within an unambiguity interval for $x_{i_0, t'}$, we may conclude $z_{t''} = x_{i_0, t''}$ in this case and then $p_{t''} < x_{i_0, t''}$. In the second case we may look at : $p_{t'} \leq \varphi_{t'' t'}(x_{i_0, t''}) < x_{i_0, t'} < 1$ (where the first inequality stems from (*) above). Again restricting the interval further (but open) we find a $z'_{t''}$ in $\Lambda_{t''}$ such that $x_{i_0, t''} < z'_{t''} < 1$ such that $\varphi_{t'' t'}(z'_{t''}) = x_{i_0, t'}$. Since we are dealing with the case $p_{t''} \neq x_{i_0, t''}$ and we are in an unambiguity interval for $p_{t'}$ it follows that $p_{t'} < \varphi_{t'' t'}(x_{i_0, t''})$. Look at : $p_{t'} < \varphi_{t'' t'}(x_{i_0, t''}) < x_{i_0, t'}$ with $\varphi_{t'' t'}(p_{t''}) = p_{t'}$ and $\varphi_{t'' t'}(z'_{t''}) = x_{i_0, t'}$; by restricting the interval (open) further if necessary we obtain the existence of $z''_{t''}$ such that, $p_{t''} < z''_{t''} < z'_{t''}$ such that $\varphi_{t'' t'}(z''_{t''}) = \varphi_{t'' t'}(x_{i_0, t''})$. Finally restricting again the $t'' \leq t'$ to vary in an unambiguity interval for $\varphi_{t'' t'}(x_{i_0, t''})$ it follows that $z''_{t''} = x_{i_0, t''}$ and hence $z''_{t''} \geq p_{t''}$ yields $x_{i_0, t''} \geq p_{t''}$ for t'' in a suitable relative open around t containing t' . This also in the case we arrive at $p_{t'} \in x_{i_0}$ or $p_{t'} \in U_t(x_{i_0})$. It follows that $\mathcal{U}_t(w) = U\{U_{i, t}, i \in \mathcal{J}\}$ establishing that arbitrary unions of opens are open. By taking $\mathcal{U}_t(1)$ we obtain the whole spectrum at t as an open too. \square

2.7 A Possible Relation to M -Theory ?

In noncommutative topology and derived point topologies the gen-topology appears naturally (and it is a classical i.e. commutative topology (cf. [VO1]). Moreover continuity in the gen-topology also appears naturally in noncommutative geometry of associative algebras but we did not ask the $\varphi_{tt'}$ in the DNT-axioms to be continuous in the gen topology. However one may prove that in general “continuity of the $\varphi_{tt'}$ in the gen-topologies of Λ_t resp. $\Lambda_{t'}$ is an observed truth ! Consequently for t' close enough to t the $\varphi_{tt'}$ is continuous with respect to the gen-topologies (cfr. [VO2]).

In the mathematical theory all Λ_t may be different and there is no reason to aim at $Sl(\Lambda_t)$ nor $\text{Spec}(\Lambda_t, I_t)$ to be invariant under t -variation. From the point

of view of Physics one may reason that only one case is important i.e. the case we see as “reality”. This being the utmost special case it is then not far-fetched to assume that the dynamic noncommutative space has as commutative shadow the abstract mathematical frame we reason in about reality. for example identified with 3 or 4 dimensional space or space-time, and also an observational mathematical frame we measure in that is $\text{Spec}(\Lambda_t, I_t)$ identified to an 11-dimensional space (for M -theory) or any other one fitting physical interpretations in some theory one chooses to believe in. An observed point in $\text{Spec}(\Lambda_t, I_t)$ is then given by a string of elements say $p_{t'} \in \Lambda_{t'}$, $t' \in J \subset I_t$ with $p_t \in \Lambda_t$ a temporal point. If $p_{t'}$ is a point then for all $t' \leq t''$, $\varphi_{t't''}(p_{t'})$ is idempotent so appears in the commutative shadow $SL(\Lambda_{t''})$. Hence an observed point in $\text{Spec}(\Lambda_t, I_t)$ appears as a string in the base space $SL(\Lambda_{t''})(t' \leq t'')$, identified with n -dimensional space but the string may “start after” t when the point was “observed”. On the other hand, the assumption that the system $(\Lambda_t, \varphi_{tt'}, T)$ is temporally pointed leads to a decomposition of every $p_{t'}$ into temporal points of $\Lambda_{t'}$ realizing it as an open of $\text{Spec}[\Lambda_{t'}, I_{t'}]$. Thus in the spectral space (identified with a certain m -dimensional space say), the observed point appears as a “string” connecting opens i.e. a possibly more dimensional string that can be thought of as a tube. The difference between the dimensions m and n has to be accounted for by the “rank” of T (e.g. if T were a group like \mathbb{R}_+^d , d would be the rank) which allowed to create the extra points in $\text{Spec}(\Lambda_t, I_t)$ when compared to $(SL(\Lambda_t))$. Note that even when the $\varphi_{tt'}$ do not necessarily define maps between $\text{Spec}(\Lambda_t, I_t)$ and $\text{Spec}(\Lambda_{t'}, I_{t'})$ or between $SL(\Lambda_t)$ and $SL(\Lambda_{t'})$, the given strings at the Λ_t -level do define sequences of elements or opens in the $\text{Spec}(\Lambda_t, I_t)$ resp. $SL(\Lambda_t)$ that may be viewed as strings, resp. tubes. Two more intriguing observations :

- i) Identifying $\text{Spec}(\Lambda_t, I_t)$ to one fixed commutative world and $SL(\Lambda_t)$ to another allows strings and tubes to be open or closed.
- ii) Only temporal points corresponding to future points can be non-idempotent, therefore all noncommutativity is due to future points and uncertainty may be seen as an effect of the possibility that the interval needed to realize the temporal point p_t by a point $p_{t''} \in \Lambda_{t''}$ for $t \leq t''$ is **larger** than the unambiguity interval for $p_{t''}$! Passing from a commutative frame $(SL(\Lambda))$ to noncommutative (dynamical) geometry and phrasing theories and calculations in $\text{Spec}(\Lambda_t, I_t)$ at the price of having to work in higher dimension seems to fit quantum theories. Of course this is at the level of mathematical formalism, for suitable interpretations within physics the physical entity connected to the notion of point in $\text{Spec}(\Lambda_t, I_t)$ should be the smallest possible, i.e. a kind of building block of everything, so that observing it as a point in the moment spaces is acceptable; that these points are mathematically described as strings or higher dimensional strings via the noncommutative geometry is a “Deus ex Machina” pointing at an unsuspected possibility of embedding M -theory in our approach. No further speculation about this here, perhaps specialists in string theory may be interested in investigating further

this formal incidence.

3 Moment Presheaves and Sheaves

Continuing the point of view put forward in the short introduction to Section 2, points or more precisely functions defined in a set theoretic spirit, should be replaced by a generalization of “germs of functions” obtained from limit constructions in classical topology terms to noncommutative structures. Thus the notion of point is replaced by an avatar of the notion “stalk” of a pregiven sheaf, more correctly when different (pre)sheaves over a noncommutative space are being considered, say with values in some nice category of objects $\underline{\mathcal{C}}$, then a “point” is a suitable limit functor on $\underline{\mathcal{C}}$ -objects generalizing the classical construction of localization (functor) at a point. Assuming that a suitable topological space and satisfactory sheaf of “functions” on it have been identified, satisfactory in the sense that it allows to study the desired geometric phenomena one is aiming at, then the notion of point via stalks should be suitable too. For example, prime ideals would be identified via stalks of the structure sheaf of a commutative Noetherian ring without having to check a primeness condition of the corresponding localization functor. More on the definition of noncommutative geometry via (localization) functors can be found in [VO1] where it is introduced as a functor geometry over a noncommutative topology also [MVO] and [VOV].

In this section we fix a category $\underline{\mathcal{C}}$ allowing limits and colimits; we might restrict to Abelian or even Grothendieck categories but that is not essential. In fact, the reader who wants to fix ideas on a concrete situation may choose to work in the category of abelian groups.

For every $t \in T$, Γ_t is a presheaf over Λ_t and for $t \leq t'$ in T there is a $\phi_{tt'} : T_t \rightarrow T_{t'}$, defined by morphisms in $\underline{\mathcal{C}}$ as follows :

- i) For $\lambda_t \in \Lambda_t$ there is a $\phi_{tt'}(\lambda_t) : \Gamma_t(\lambda_t) \rightarrow \Gamma_{t'}(\phi_{tt'}(\lambda_t))$
- ii) for $\mu_t \leq \lambda_t$ in Λ_t we have commutative diagrammes in $\underline{\mathcal{C}}$:

$$\begin{array}{ccc} \Gamma_t(\lambda_t) & \xrightarrow{\phi_{tt'}(\lambda_t)} & \Gamma_{t'}(\phi_{tt'}(\lambda_t)) \\ \downarrow \rho_{\lambda_t, \mu_t}^t & & \downarrow \rho_{\lambda_{t'}, \mu_{t'}}^{t'} \\ \Gamma_t(\mu_t) & \xrightarrow{\phi_{tt'}(\mu_t)} & \Gamma_{t'}(\phi_{tt'}(\mu_t)) \end{array}$$

where we have written $\lambda_{t'}, \mu_{t'}$ for $\phi_{tt'}(\lambda_t)$, resp. $\phi_{tt'}(\mu_t)$ and $\rho_{\lambda_{t'}, \mu_{t'}}^{t'}$ for the restriction morphism of $\Gamma_{t'}$.

- iii) $\phi_{tt}(\lambda_t) = I_{\Gamma_t(\lambda_t)}$ for all $t \in T$, and for $t \leq t'$ and let $t' \leq t''$ we have $\phi_{t't''}(\phi_{tt'}(T_t(\lambda_t))) = \phi_{tt''}(T_t(\lambda_t))$ for all $\lambda_t \in \Lambda_t$.

The system $\{\Gamma_t, \phi_{tt'}, T\}$ is called a (global) dynamical presheaf over the DNT $\{\Lambda_t, \phi_{tt'}, T\}$.

Since sheaves on a noncommutative topology do not form a topos it is a problem to define a suitable sheafification i.e. : can a presheaf Γ on Λ be sheafified to a sheaf $\underline{g}\Gamma$ on the same Λ via a suitable notion of “stalk”, then allowing interpretations in terms of “points” ? In fact, the axioms of DNT allow to give a solution to the sheafification problem at the price that the sheaf $\underline{g}\Gamma_t$ has to be constructed over $\text{Spec}(\Lambda_t, I_t)$!

From hereon we let $\underline{\Lambda} = \{\Lambda_t, \varphi_{tt'}, T\}$ be a temporally pointed system which is a space continuum. We refer to $Y_t = \text{Spec}(\Lambda_t, I_t)$ with its spectral topology as the **moment space** at $t \in T$.

For $p_{t'} \in Y_t$ we may calculate (in $\underline{\mathcal{C}}$): $\Gamma_{t', p_{t'}} = \varinjlim \Gamma_{t'}(x_t)$ where \varinjlim is over $x_t \in \Lambda_{t'}$ such that $p_{t'} \leq x_t$, and where $x = (\dots, x_t, \dots)$ is t -accessible, in fact we have $p_{t'} \in x$. In the foregoing we did not demand $\underline{\Lambda}$ to derive from a system \underline{X} and passing from X_t to Λ_t as a generalized Stone space or pattern topology via $C(X_t)$. We preferred not to dwell upon the formal comparison of dynamical theories for the X_t and the Λ_t . In order to keep trace of a possible original X_t if it was considered in the construction of Λ_t one may if desired use the following.

3.1 Definition

We say that $u_t \in \Lambda_t$ is **classical** if $u_t = [\chi_t]$ for $\chi_t \in X_t$. If u_t is classical then there is an open interval containing t in T , say L , such that for every $t' \in L$ we have that $u_{t'}$ is classical, where for $t \leq t'$ we have $u_{t'} = \varphi_{tt'}(u_t)$ and for $t' \leq t$, $u_{t'}$ is a suitably chosen representative for u_t , $\varphi_{t't}(u_{t'}) = u_t$. Restricting further to an unambiguity interval of u_t , the representations $u_{t'}$ for t' in that interval are unique. Since points of X_t are by definition elements in $C(X_t)$, the filter in Λ_t defined by that point has a cofinal directed subset of classical elements. When in $\{\Lambda_t, \varphi_{tt'}, T\}$ we restrict attention to classical x , i.e. every x_t classical Λ_t , then we say that we look at a **traditional system**.

3.2 Lemma

For a traditional space continuum with dynamical presheaf $\{\Gamma_t, \phi_{tt'}, T\}$, the stalk for $p_{t'} \in Y_t$ of $\Gamma_{t'}$ is exactly $\Gamma_{t', p_{t'}}$ as defined above.

Proof In calculating $\varinjlim \{\Gamma_{t'}(u_{t'}), p_{t'} \leq u_{t'}\}$ we may restrict to classical $u_{t'}$ in $\Lambda_{t'}$. It now suffices to establish the existence of a t -accessible y such that $p_{t'} \in y$ and $y_{t'} \leq u_{t'}$. From $p_{t'} \in U_t(x)$ we obtain $(\dots, x_{t'}, \dots)$ with a relative open T -interval $J, t' \in J$, such that $p_{t''} \leq x_{t''}$ for every $t'' \in J$. Since $u_{t'}$ and $x_{t'}$ are classical, so is $u_{t'} \wedge x_{t'}$ and moreover $p_{t'} \leq u_{t'} \wedge x_{t'}$ because $p_{t'}$ is a point in $\Lambda_{t'}$ (hence idempotent !). Let J_1 be an open T -interval containing t' such that $u_{t'} \wedge x_{t'}$ has a representative $u_{t''}$ in $\Lambda_{t''}$ such that $\varphi_{t''t'} = u_{t'} \wedge x_{t'}$. Since $x_{t'} \neq u_{t'} \wedge x_{t'}$ may be assumed (otherwise put $y = x$) we arrive at $p_{t''} \leq u_{t''} < x_{t''}$. Using the intersection of J_1 and the interval around t' allowing to select classical $u_{t''}$, call this interval J_2 , we put $y_{t''} = u_{t''}$ for $t'' \leq t'$ in J_2 and $y_{t_1} = x_{t_1}$ for $t' < t_1$ in J . Then y is t -accessible with respect to the

relative open T -interval around t just defined : we have $y_{t'} \leq u_{t'}$ and $p_{t'} \in y$. Consequently : $\lim_{\substack{\longrightarrow \\ p_{t'} \leq u_{t'}}} \Gamma_{t'}(u_{t'}) = \lim_{\substack{\longrightarrow \\ p_{t'} \in x}} \Gamma_{t'}(x_{t'})$. \square

In the sequel we assume objects in $\underline{\mathcal{C}}$ are at least sets but let us restrict to abelian groups. Again let $\{\Gamma_t, \phi_{tt'}, T\}$ be a dynamical presheaf over a traditional space continuum. On Y_t we define a presheaf, with respect to the spectral topology, by taking for $\mathcal{P}(U_t(x))$ the abelian group in $\coprod_{t' \in I_t} \Gamma_{t'}(x_{t'})$ formed by strings over $\text{sup}(x) = \{t' \in I_t, x_{t'} \neq 0\}$. i.e. $\{\gamma_{t'}, t' \in \text{sup}(x), \phi_{t''t'}(\gamma_{t''}) = \gamma_{t'}$ for $t'' \leq t'$ in $\text{sup}(x)\}$. Let us write $x < y$ if $x_{t'} \leq y_{t'}$ for all t' in I_t , in particular $x < y$ means $\text{sup}(x) \subset \text{sup}(y)$. In sheaf theory one usually omits the empty set, so here the $0 \in \Lambda_t$ at every t , and it makes sense to do that here as well. However one may define at every $t' \in T$, $\Gamma_{t'}(0) = \lim_{\longrightarrow} \{\Gamma_{t'}(x_{t'}), x_{t'} \text{ classical in } \Lambda_{t'}\}$ and all statements made in the sequel will remain consistent.

If $x < y$ then we have restriction morphisms $\rho_{y_{t'}, x_{t'}}^{t'} : \Gamma_{t'}(y_{t'}) \rightarrow \Gamma_{t'}(x_{t'})$. Commutativity of the diagrams in the beginning of the section yield corresponding morphisms on the strings over the respective supports : $\rho_{y,x} : \mathcal{P}(U_t(y)) \rightarrow \mathcal{P}(U_t(x))$. For a point $p_{t'}$ we let $\eta(p_{t'})$ be the set of $U_t(x)$ such that we have $p_{t'} \in U_t(x)$ i.e. $p_{t'} \in x$; in particular $t' \in J_x$ where J_x is the relative open around t in the definition of x and consequently : $t' \in \cap \{\text{sup}(x), \eta(p_{t'}) \text{ contains } U_t(x)\}$. For the dynamical sheaf theory we may want to impose coherence conditions on the system assuming some relations between $\Gamma_{t''}$ and $\Gamma_{t'}$ if t' and t'' are close enough in T . We shall restrict here to only one extra assumption, in some sense dual to the unambiguity interval assumption for the underlying DNT.

3.3 Definition

The dynamical presheaf $\{\Gamma_t, \phi_{tt'}, T\}$ on a traditional space continuum is locally temporally flabby at $t \in T$ if for t -accessible x such that $p_{t'} \in x$ and $s_{t'} \in \Gamma_{t'}(x_{t'})$ there exists a t -accessible $y < x$ with $p_{t'} \in y$ and a string $\bar{s} \in \mathcal{P}(U_t(y))$ such that $\bar{s}_{t'} = \rho_{x_{t'}, y_{t'}}^{t'}(s_{t'})$.

3.4 Theorem

To a dynamical presheaf on a traditional space continuum there corresponds for every $t \in T$ a presheaf \mathcal{P}_t on the moment space $\text{Spec}(\Lambda_t, I_t)$ with its spectral topology given by the $U_t(x)$ for t accessible x . In case all $\Gamma_{t'}, t' \in I_t$, are separated presheaves then \mathcal{P}_t is separated too. The sheafification $\underline{\underline{\mathcal{P}}}_t$ of \mathcal{P}_t on the moment space $\text{Spec}(\Lambda_t, I_t) = Y_t$ is called the **moment sheaf of spectral sheaf** at $t \in T$. In case the dynamical presheaf is locally temporally flabby (LTF) then for any point $p_{t'} \in Y_t$ the stalk $\mathcal{P}_{t, p_{t'}}$ may be identified with $\Gamma_{t', p_{t'}}$.

Proof At every $t \in T$, \mathcal{P}_t is the spectral presheaf constructed on $\text{Spec}(\Lambda_t, I_t)$ with its spectral topology. Now suppose all Γ_t are separated presheaves and look at a finite cover $U_t(x) = U_t(x_1) \cup \dots \cup U_t(x_n)$ and a $\gamma \in \Gamma_t(U_t(x))$ such that for $i = 1, \dots, n$, $\rho_{x, x_i}(\gamma) = 0$. We have seen before that the union $U_t(x_1) \cup$

$\dots \cup U_t(x_n)$ corresponds to the t accessible element $x_1 \wedge \dots \wedge x_n$ obtained as the string over $\sup(x_1) \wedge \dots \wedge \sup(x_n)$ given by the $x_{1,t'} \cup \dots \cup x_{n,t'}$ in $\Lambda_{t'}$. For all $t' \in \sup(x)$ we obtain, in view of the compatibility diagrams for restrictions and $\phi_{t',t''}, t' \leq t'' : \rho_{x_{t'}, x_{i,t'}}^{t'}(\gamma_{t'}) = 0$, for $i = 1, \dots, n$. The assumed separatedness of $\Gamma_{t'}$, for all t' then leads to $\gamma_{t'} = 0$ for all $t' \in \sup(x)$ and therefore $\gamma = 0$ as a string over $\sup(x)$. Consequently \mathcal{P}_t is separated, for all $t \in T$. In order to calculate the stalk at $p_{t'} \in \text{Spec}(\Lambda_t, I_t)$ for \mathcal{P}_t we have to calculate : $\lim_{\substack{\longrightarrow \\ p_{t'} \in x}} \Gamma_t(U_t(x)) = E_{t'}$.

Starting with $p_{t'} \in x$ for some t -accessible x we have a representative $\gamma_x \in \Gamma_t(U_t(x))$ being a string over $\sup(x)$ and the latter containing a relative open $J(x)$ around t containing t' . So an element $e_{t'}$ in $E_{t'}$ may be viewed as given by a direct family $\{\gamma_x, p_{t'} \in x, \rho_{x,y}(\gamma_x) = \gamma_y \text{ for } y < x\}$. At t' , which is in $\sup(x)$ for all x appearing in the foregoing family (as $U_t(x)$ varies over $\eta(p_{t'})$), we obtain $\{(\gamma_x)_{t'}, p_{t'} \leq x_{t'}, \rho_{x_{t'}, y_{t'}}^{t'}((\gamma_x)_{t'}) = (\gamma_y)_{t'} \text{ which defines an element of } \Gamma_{t', p_{t'}}$, say $\bar{e}_{t'}$. We have a well-defined map $\pi_{t'} : E_{t'} \rightarrow \Gamma_{t', p_{t'}}, e_{t'} \mapsto \bar{e}_{t'}$. Without further assumptions we therefore arrive at a sheaf $\underline{\mathcal{P}}_t$ with stalk $E_{t'}$ at $p_{t'}$ and a presheaf map $\mathcal{P}_t \rightarrow \underline{\mathcal{P}}_t$ which is “injective” in case all $\Gamma_{t''}$ are separated. Now we have to make use of the locally temporally flabbiness (LTF). Look at a germ $s_{t'} \in (\Gamma_{t'})_{p_{t'}}$. In view of Lemma 3.2. there exists a t -accessible x such that $s_{t'} \in \Gamma_{t'}((x_{t'}))$ with $p_{t'} \in x$, in particular $p_{t'} \leq x_{t'}$.

The LTF-condition allows to select a t -accessible $y < x$ with $p_{t'} \in y$ together with a string, $\vec{s}(y) \in \mathcal{P}(U_t(y))$ such that $\vec{s}_{t'}(y) = \rho_{x_{t'}, y_{t'}}^{t'}(s_{t'})$. The element $e_{t'}$ in $E_{t'}$ defined by the directed family obtained by taking restrictions of $\vec{s}_{t'}(y)$ has $\bar{e}_{t'}$ exactly $s_{t'}$ (note that t' supports all the restrictions of $\vec{s}_{t'}(y)$ because y varies in $\eta(p_{t'})$). Thus $\pi_{t'} : E_{t'} \rightarrow \Gamma_{t', p_{t'}}$ is epimorphic. If $e_{t'}$ and $e'_{t'}$ have the same image under $\pi_{t'}$ then there is a t -accessible y such that $e_{t'} - e'_{t'}$ is represented by the zero-string over $\sup(y)$; in fact this follows by taking $s_{t'} = 0$ in the foregoing. Leading to a t -accessible y as above that may be restricted to a t -accessible y' defined by taking for $\sup(y')$ the relative open J containing t' in the support of y where $\vec{s}_{t'}(y) = 0$. Therefore, $\pi_{t'}$ is also injective. \square

Can one avoid a condition like LTF in the foregoing theorem ? It seems that the idea of “germ” appearing in the notion of stalks spatially needs an extension in the temporal direction, so probably some condition close to LTF is really necessary here.

4 Some Remarks on Spectral families and Observables

Let Γ be any totally ordered abelian group. On a noncommutative topology Λ we define a Γ -filtration by a family $\{\lambda_\alpha, \alpha \in \Gamma\}$ such that for $\alpha \leq \beta$ in Γ , $\lambda_\alpha \leq \lambda_\beta$ in Λ and $\vee\{\lambda_\alpha, \alpha \in \Gamma\} = 1$. A Γ -filtration is said to be separated if from $\gamma = \inf\{\gamma_\alpha, \alpha \in \mathcal{A}\}$ in Γ it follows that $\lambda_\gamma = \wedge\{\lambda_\alpha, \alpha \in \mathcal{A}\}$ and $0 = \wedge\{\lambda_\gamma, \gamma \in \Gamma\}$. A Γ -spectral family in Λ is just a separated Γ -filtration, it

may be seen as a map $F : \Gamma \rightarrow \Lambda, \gamma \mapsto \lambda_\gamma$, where F is a poset map satisfying the separatedness condition. Note that by definition the order in $\wedge\{\lambda_\gamma, \gamma \in \Gamma\}$ does not matter but λ_{γ_α} need not be idempotent in Λ . Taking $\Gamma = \mathbb{R}_+$ and $\Lambda = L(H)$ the lattice of a Hilbert space H , we recover the usual notion of a spectral family. We say that a Γ -spectral family on Λ is idempotent if $\lambda_\gamma \in \text{id}_\Lambda(\Lambda)$ for every $\gamma \in \Gamma$.

Observation If Γ is indiscrete, i.e. for all $\gamma \in \Gamma, \gamma = \inf\{\tau, \gamma < \tau\}$ (example $\Gamma = \mathbb{R}_+^n$), then every Γ -spectral family is idempotent.

4.1 Proposition

- i) Let us consider a Γ -spectral family on Λ , then for $\gamma, \tau \in \Gamma : \lambda_\gamma \wedge \lambda_\tau = \lambda_\delta \wedge \lambda_\gamma = \lambda_\delta$, where $\delta = \min\{\tau, \gamma\}$.
- ii) If the Γ -spectral family is idempotent then for $\gamma, \tau \in \Gamma, \lambda_\gamma \wedge \lambda_\tau = \lambda_\tau \wedge \lambda_\gamma$ and the Γ -spectral family on Γ is in fact a Γ -spectral family of the commutative shadow $SL(\Lambda)$.

Proof Easy enough. □

A filtration F on a noncommutative Λ is said to be **right bounded** if $\lambda_\gamma = 1$ for some $\gamma \in \Gamma$, F is **left bounded** if $\lambda_\delta = 0$ for some $\delta \in \Gamma$.

For a right bounded Γ -filtration $F : \Gamma \rightarrow \Lambda$ we may define for every $\mu \in \Lambda$ the induced filtration $F|_\mu : \Gamma \rightarrow \Lambda(\mu)$ where we use $\mu = 1_{\wedge(\mu)}$, $\Lambda(\mu) = \{\lambda \in \Lambda, \lambda \leq \mu\}$. Note that $F|_\mu$ need **not** be separated whenever F is, indeed if $\delta = \inf\{\delta_\alpha, \alpha \in \mathcal{A}\}$ in Γ then $\lambda_\delta = \wedge\{\lambda_{\delta_\alpha}, \alpha \in \mathcal{A}\}$ in Γ then $\lambda_\delta = \wedge\{\lambda_{\delta_\alpha}, \alpha \in \mathcal{A}\}$ but $\mu \wedge \lambda_\delta$, and $\wedge\{\mu \wedge \lambda_\alpha, \alpha \in \mathcal{A}\}$ need not be lequal in general.

4.2 Proposition

If F defines a right bounded Γ -spectral family on Λ then $F|_\mu$ is a spectral family of $\Lambda(\mu)$ in each of the following cases :

- a. $\mu \in \text{id}_\Lambda(\Lambda)$ and μ commutes with all $\lambda_\alpha, \alpha \in \Gamma$.
- b. $\mu \wedge \lambda_\alpha$ is idempotent for each $\alpha \in \Gamma$.

Proof Easy and straightforward. □

An element μ with property a. as above is called an **F -centralizer** of Λ .

4.3 Corollary

In case Λ is a lattice then for every $\mu \in \Lambda$ a right bounded Γ -spectral family of Λ induces a right bounded spectral family on $\Lambda(\mu)$.

Let F be a Γ -spectral family on a noncommutative Λ . To $\lambda \in \Lambda$ associate $\sigma(\lambda) \in \Gamma \cup \{\infty\}$ where $\sigma(\lambda) = \inf\{\gamma, \lambda \leq \lambda_\gamma\}$ and we agree to write $\inf \emptyset = \infty$.

The map $\sigma : \Lambda \rightarrow \Gamma \cup \{\infty\}$ is a generalization of the principal symbol map in the theory of filtered rings and their associated graded rings. We refer to σ as the **observable function** of F .

$$\begin{aligned} \text{Clearly } : \sigma(\lambda \wedge \mu) &\leq \min\{\sigma(\lambda), \sigma(\mu)\} \\ \sigma(\lambda \vee \mu) &\leq \max\{\sigma(\lambda), \sigma(\mu)\} \end{aligned}$$

The **domain of σ** is $\cup\{[0, \lambda_\gamma], \gamma \in \Gamma\}$ and observe that $\vee\{\lambda_\gamma, \gamma \in \Gamma\} = 1$ does not imply that $D(\sigma) = \Lambda$ (can even be checked for $\Gamma = \mathbb{R}_+, \Lambda = L(H)$).

If $F : T \rightarrow L(H)$ is a Γ -spectral family and $V \subset H$ a linear subspace, then we may define $\gamma_V \in \Gamma$, $\gamma_V = \inf\{\gamma \in \Gamma, \subset L(H)_\gamma\}$, again putting $\inf \emptyset = \infty$. The map $\rho : L(H) \rightarrow \Gamma \cup \{\infty\}, U \mapsto \rho(U) = \gamma_U$ is well-defined. One easily verifies for \cup and \vee in H :

$$\begin{aligned} \rho(U + V) &\leq \max\{\rho(U), \rho(V)\} \\ \rho(U \cap V) &\leq \min\{\rho(U), \rho(V)\} \end{aligned}$$

The function ρ defines a $\underline{\rho}$ defined on H by putting $\rho(x) = \rho(\mathbb{C}x)$. We denote $\underline{\rho}$ again by ρ and call it the **pseudo-place of the Γ -spectral family**. Then any Γ -spectral family defines a function on the projective Hilbert space $\mathbb{P}(H)$ described on the lines in H by $\bar{\rho} : \mathbb{P}(H) \rightarrow \Gamma, \underline{\mathbb{C}v} \mapsto \rho(\mathbb{C}v)$, where we wrote $\underline{\mathbb{C}v}$ for $\mathbb{C}v$ as an object in $\mathbb{P}(H)$.

The pseudo-place aspect of ρ translates to $\bar{\rho}$ in the following sense : $\mathbb{C}w \subset \mathbb{C}v + \mathbb{C}u$ we have $\underline{\rho}(\mathbb{C}w) \leq \max\{\bar{\rho}(\underline{\mathbb{C}v}), \bar{\rho}(\underline{\mathbb{C}u})\}$.

A linear subspace $U \subset H$ such that $\mathbb{P}(U) \subset \bar{\rho}^{-1}(] - \infty, \gamma])$ allows for $u \neq 0$ in $U : \rho(\mathbb{C}u) \leq \gamma$ i.e. $u \in L(H)_\gamma$. Hence, the largest U in H such that $\mathbb{P}(U)$ is in $\bar{\rho}^{-1}(] - \infty, \gamma])$ is exactly $L(H)_\gamma$; this means that the filtration F may be reconstructed from the knowledge of $\bar{\rho}$. One easily recovers the classical result that maximal abelian Von Neumann regular subalgebras of $\mathcal{L}(H)$ correspond bijectively to maximal distributive lattices in $L(H)$. Since any Γ -spectral family is a directed set in Λ it defines an element of $C(\Lambda)$ which we call a **Γ -point**. The set of Γ -points of Λ is denoted $[\Gamma] \subset C(\Lambda)$. We may for example think of $[\mathbb{R}] \subset C(L(H))$ as being identified via the Riemann-Stieltjens integral to the set of self-adjoint operators on H .

Let $\sigma : \Lambda \rightarrow \Gamma \cup \{\infty\}$ be the observable functions of a Γ -spectral family on Λ defined by $F : \Gamma \rightarrow \Lambda$. Put $\mathcal{F} : C(\Lambda) \rightarrow \Gamma \cup \{\infty\}, [A] \mapsto \inf\{\gamma \in \Gamma, \lambda_\gamma \in \overline{A}\}$, \overline{A} the filter of A . Then $\hat{\sigma}$ is the observable corresponding to the Γ -filtration on $C(\Lambda)$ defined by $[A]_\gamma$, where for $\gamma \in \Gamma, [A]_\gamma$ is the class of the smallest filter containing λ_γ i.e. the filter $\{\mu \in \Lambda, \lambda_\gamma \leq \mu\}$. This is clearly a Γ -spectral family because in fact $[A]_\gamma < [\lambda_\gamma]$. We define $[\Gamma] \cap \text{Sp}(\Lambda) = \Gamma - \text{Sp}(\Lambda)$, $[\Gamma] \cap QSp(\Lambda) = \Gamma - QSp(\Lambda)$ and similarly with p replaced by P when $\text{Id}_\Lambda(C[\Lambda])$ is considered instead of $\text{id}_\Lambda(C(\Lambda_-))$ (see section 1).

In view of Proposition 4.1.i. a Γ -spectral family is contained in a sublattice (that is with commutative \wedge) of the noncommutative Λ , in fact $\{\lambda_\gamma, \gamma \in \Gamma\}$ is such a sublattice. If $Ab(\Lambda)$ is the set of maximal commutative sublattices of Λ then every Γ -spectral family in Λ is a Γ -spectral family in some $B \in Ab(\Lambda)$ (B

refers to Boulean sector in case $\Gamma = \mathbb{R}_+, \Lambda = L(H)$). The above remarks may be seen as a generalization of the result concerning maximal commutative Von Neumann regular subalgebras in $\mathcal{L}(H)$ quoted above.

Γ -spectral families may be defined on the moment spaces $\text{Spec}(\Lambda_t, T_t)$ in exactly the way described above as filtrations $\{U_t(x_\gamma), \gamma \in \Gamma\}$, where each x_γ is t -accessible, defining a separated Γ -filtration. For $t'' \in I_t$ we may look at $V_t(x_\gamma) = \{p_{t''}, p_{t''} \in U_t(x_\gamma)\}$, again $p_{t''} = \varphi_{t't''}(p_{t'})$ or $\varphi_{t't''}(p_{t'}) = p_{t''}$ depending whether $t' \leq t''$ or $t'' \leq t'$. The family $\{V_t(x_\gamma), \gamma \in \Gamma\}$ need not (!) be a Γ -spectral family at $t'' \in T$. A stronger notion of **dynamical spectral family** may be obtained by demanding the existence of stringwise spectral families in a relative open T -interval J around t . Then indeed at $t'' \in J \subset I(t)$ such a stringwise Γ -spectral family induces a γ -spectral family in $\Lambda_{t''}$ but not immediately on $\text{Spec}(\Lambda_{t''}, I_{t''})$ unless a more stringent relation is put on $I_{t''}$ and its comparison with respect to I_t . We just point out the interesting problems arising with respect to observables when passing to moment spaces but this is work in progress.

References

- [VO1] F. Van Oystaeyen, *Algebraic Geometry for Associative Algebras*, M. Dekker, Math. Monographs, Vol. 232, New York, 2000.
- [Co] A. Connes, *C*-Algeèbres et Géométries Differentielle*, C. R. Acad. Sc. Paris, 290, 1980, 599-604.
- [CoD] A. Connes, M. Dubois-Violette, *Noncommutative Finite Dimensional Manifolds I, Spherical Manifolds and related Examples*, Notes IHES, MO1, 32, 2001.
- [A.T.V.] M. Artin, J. Tate, M. Van den Bergh, *Modules over Regular Algebras of Dimension 3*, Invent. Math. 106, 1991, 335-388.
- [VdB] M. Van den Bergh, *Blowing up of Noncommutative Smooth Surfaces*, Mem. Amer. Math. Soc., 154, 2001, no. 734.
- [Ko] M. Kontsevich, *Deformation, Quantization of Poissin Manifolds*, q-alg, 9709040, 1997.
- [VO.2] F. Van Oystaeyen, *Virtual Topology and Functor Geometry*, lecture Notes UA, Submitted.
- [MVO] D. Murdoch, F. Van Oystaeyen, *Noncommutative Localization and Sheaves*, J. of Algebra, 35, 1975, 500-525.
- [S] J.-P. Serre, *Faisceaux Algébriques Cohérents*, Ann. Math. 61, 1955, 197-278.
- [VOV] F. Van Oystaeyen, A. Verschoren, *Noncommutative Algebraic Geometry*, LNM 887, Springer Verlag, Berlin, 1981.